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ON THE STABILITY AND POSTBUCKLING BEHAVIOUR OF MULTILAYERED SANDWICH-TYPE PLATES

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Abstract—This paper gives the governing equations and natural boundary conditions for the stability of multilayered sandwich-type rectangular flat plates with orthotropic (constructionally orthotropic) hard and transversally isotropic soft layers. It shows the solution of these equations and the problem of initial postbuckling behaviour for the regularly multilayered plates with Navier-type boundary conditions.

1. INTRODUCTION

The theories and analyses for the solution of different problems of multilayered media (including beams, plates and shells) depend on the objectives of the investigations. The actual layouts, the material characteristics, the loading and boundary conditions determine which of either theories or models shall be or may be used. These investigations in general must be considered as a three-dimensional problem, but—depending on the characteristic wave lengths, belonging to the "changing" of the strain/stress state of the multilayered media/construction—major emphasis has been concentrated on the approach solutions. Examples of the characteristic theories and models are given in all the references cited.

The sandwich-type constructions (beams, plates, shells) can be characterized by the very different material characteristics of the neighbouring layers. In such constructions the "hard" layers are usually modelled as thin plates/shells obeying the Kirchhoff–Love hypothesis, and for the "soft" layers the antiplane shear strains and stresses are characteristic, while the model of "transversally soft" layers also includes the effects of the antiplane normal deformations (Bolotin, 1965a,b).

Using the Bolotin's model and the results of Pomázi and Moskalenko (1967) the stability and initial postbuckling behaviour of the regularly multilayered plates with orthotropic hard and transversally isotropic soft layers have been investigated by the author (Pomázi, 1966, 1978, 1982).

The present report gives a review of the formulation of the mechanical/mathematical models of the stability and initial postbuckling behaviour of multilayered sandwich-type plates with orthotropic (possibly constructionally orthotropic) hard and transversally isotropic soft layers. The corresponding governing equations and boundary conditions are derived and the formulation of the problem of initial postbuckling behaviour of the plate are given. The formulated equations and tasks—as examples—are shown for the regularly multilayered rectangular plate with Navier-type boundary conditions.

2. HARD AND SOFT LAYERS: MATERIAL LAWS

Let us consider a rectangular multilayered sandwich-type flat plate of side lengths a and b lying in the $0 \leqslant x \leqslant a$, $0 \leqslant y \leqslant b$ domain of Descartes' system of coordinates, composd of n "hard" and $(n-1)$ "transversally soft" layers. Let us note the hard layers beginning from the middle layer of the plate as follows:

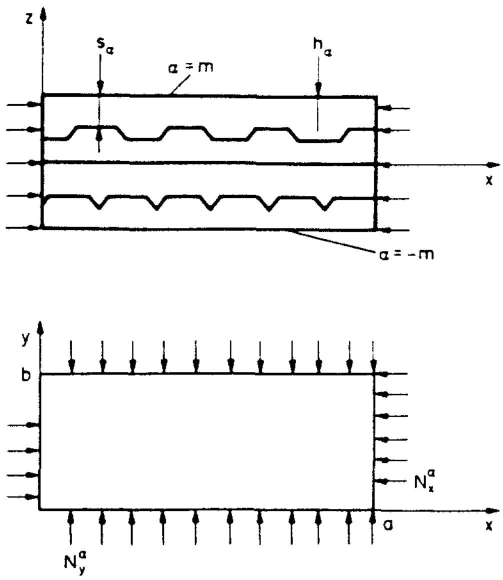

Fig. 1. The construction and loading of the plate.

$$\alpha = \pm 1, \pm 3, \ldots, m \qquad \text{if } n \text{ is even,}$$

$$\alpha = 0, \pm 2, \pm 4, \ldots, \pm m \quad \text{if } n \text{ is odd.}$$

Here in both cases $m = n-1$, and $\Delta\alpha = 2$ is the labelling's difference. The α soft layers occur between the α and $(\alpha+\Delta\alpha)$ hard layers. Let the plate be loaded on the edges only, so that in each hard layer the membrane forces N_x^α, N_y^α and N_{xy}^α are operating (Fig. 1).

It is supposed that the deformation and stress state of the plate can be described by the Cauchy deformation tensor, and that the material of the layers is elastic and orthotropic; thus Hooke's law is valid. Based on Bolotin's (1965) investigations the common assumptions from the three-layered sandwich plate theory—in addition to the effects of the Poisson ratios in the soft layers, which could be significant in the task of the stability (Pomázi and Moskalenko, 1967)—are taken into consideration.

In the frame of these assumptions for the hard layers the Kirchhoff–Love Law is valid, but in the transversally soft layers the antiplane deformations ($\gamma_{xz}, \gamma_{yz}, \varepsilon_z$) and stresses are characteristic and constant across the thickness of these layers.

With $x \equiv 1$, $y \equiv 2$, $z \equiv 3$ being the axes of orthotropy, using the common symbols for the material characteristics and deformation/stress tensor components, and based on the above assumptions, the inverse Hooke's Law for the transversally isotropic materials of the layers is in the form:

$$\underset{\cdot}{\sigma} = \underline{B} \cdot \underset{\cdot}{\varepsilon} \tag{1}$$

where *for the hard layers*:

$$b_{11} = \frac{\bar{E}_1}{\bar{\nu}}, \quad b_{12} = \frac{\bar{\nu}_{12}\bar{E}_1}{\bar{\nu}}, \quad b_{22} = \frac{\bar{E}_2}{\bar{\nu}}, \quad b_{66} = G_{12}, \quad \bar{\nu} = 1 - \bar{\nu}_{12}\bar{\nu}_{21},$$

and *for the soft layers*:

$$b_{44} = b_{55} = G, \quad b_{33} = E' \frac{1-\nu}{1-\nu-2\nu_0\nu'}.$$

Here:

$$E' = E_3, \quad G = G_{13} = G_{23}, \quad \nu = \nu_{12} = \nu_{21}, \quad \nu_0 = \nu_{13} = \nu_{23}, \quad \nu' = \nu_{31} = \nu_{32}.$$

These assumptions also mean that the expressions of the deformation energy densities for the hard and transversally soft layers—missing the second order terms—have the forms:

$$\begin{aligned} \mathrm{d}\bar{U} &= \tfrac{1}{2}(\bar{\sigma}_x\bar{\varepsilon}_x + \bar{\sigma}_y\bar{\varepsilon}_y + \bar{\tau}_{xy}\bar{\gamma}_{xy}), \\ \mathrm{d}U &= \tfrac{1}{2}(\bar{\tau}_{xz}\bar{\gamma}_{xz} + \bar{\tau}_{yz}\bar{\gamma}_{yz} + \bar{\sigma}_z\bar{\varepsilon}_z). \end{aligned} \tag{2}$$

Here the term $\bar{\sigma}_z\bar{\varepsilon}_z$ corresponds to the effect of antiplane tension/compression of the transversally soft layer. If this term is negligibly small compared with the others, then the layer is named as simply "soft". More sophisticated analysis of the characters of the layers—based on the deformation energy function—is given by Bolotin (1965) and Pomázi and Moskalenko (1967).

3. GENERALIZED CONSTITUTIVE EQUATIONS

In real multilayered sandwich-type constructions the "hard" layers could have reinforcements or they could be reinforced by foldings or otherwise, and the "soft" layers play the role of fillers (cores) between the hard layers. In these cases the constitutive equations for the layers should be "generalized" by "smoothing" of the stiffness characteristics by the thickness of the layer using the suppositions according to the strain/stress field of these layers. This method was used, for example, in the task of stability of a sandwich plate with bent hard faces (Pomázi, 1980, 1990).

Using the common definitions and symbols for the internal membrane forces and moments, based on the validity of the Kirchhoff–Love Law, after integration of Hooke's law or from the equivalence of deformation energy of the stiffened (bent) and flat layer with uniform thickness of h_s, for each *hard layer* as orthotropic plate the equivalent stiffness characteristics and the generalized Hooke's Law—connecting the internal forces and strains in the middle plane of the hard layer—can be determind in the form:

$$\begin{bmatrix} \underset{\sim}{N} \\ \underset{\sim}{M} \end{bmatrix} = \begin{bmatrix} \underset{\sim}{C} & \underset{\sim}{K} \\ \underset{\sim}{K} & \underset{\sim}{D} \end{bmatrix} \begin{bmatrix} \underset{\sim}{\varepsilon} \\ \underset{\sim}{\kappa} \end{bmatrix}. \tag{3}$$

Here:

$$N = \begin{bmatrix} N_x \\ N_y \\ T \end{bmatrix}, \quad M = \begin{bmatrix} M_x \\ M_y \\ H \end{bmatrix}, \quad \underset{\sim}{\varepsilon} = \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma \end{bmatrix}, \quad \underset{\sim}{\kappa} = \begin{bmatrix} \kappa_x \\ \kappa_y \\ 2\chi \end{bmatrix}$$

are the vectors of membrane forces, internal moments, strains and curvatures;

$$C_{ik}, \quad K_{ik}, \quad D_{ik}$$

are the stiffness characteristics; and the stiffness matrices have the forms:

$$\underset{\sim}{C} = \begin{bmatrix} C_{11} & C_{12} & 0 \\ C_{12} & C_{22} & 0 \\ 0 & 0 & C_{66} \end{bmatrix}, \quad \underset{\sim}{K} = \begin{bmatrix} K_{11} & K_{12} & 0 \\ K_{12} & K_{22} & 0 \\ 0 & 0 & K_{66} \end{bmatrix}, \quad \underset{\sim}{D} = \begin{bmatrix} D_{11} & D_{12} & 0 \\ D_{12} & D_{22} & 0 \\ 0 & 0 & D_{66} \end{bmatrix}.$$

The matrix $\underset{\sim}{K}$ characterizes the coupling effect between stretching and bending, which is significant for the constructionally orthotropic plates.

Corresponding to the assumptions for the stresses and deformations in the soft layers, which were assumed to be constant along the layers, for the generalized constitutive equation of the *transversally soft layers* with uniform thickness s we have:

$$\tilde{N} = \begin{bmatrix} \tilde{T}_{xz} \\ \tilde{T}_{yz} \\ \tilde{N}_z \end{bmatrix} = \begin{bmatrix} \tilde{C}_{13} & 0 & 0 \\ 0 & \tilde{C}_{23} & 0 \\ 0 & 0 & \tilde{C}_{33} \end{bmatrix} \begin{bmatrix} \tilde{\gamma}_{xz} \\ \tilde{\gamma}_{yz} \\ \tilde{\varepsilon}_z \end{bmatrix} = \underline{\tilde{C}} \cdot \tilde{\varepsilon} \tag{4}$$

where the stiffness characteristics are:

$$\tilde{C}_{13} = \tilde{C}_{23} = s^2 B, \quad \tilde{C}_{33} = s^2 R.$$

Here:

$$B = \frac{\tilde{G}}{s}, \quad R = \frac{\tilde{E}_z}{s},$$

where $\tilde{G}$ and $\tilde{E}_z$ are the average of the shear and the Young's moduli in normal direction to the plate of the soft layer.

4. STRAIN AND STRESS FIELDS. DEFORMATION ENERGY DENSITIES

The mechanical model of the multilayered sandwich-type plate built up in the previous section is continuous by the in-plane coordinates x, y and discrete by the coordinate z, perpendicular to the plate (by α). The displacements in the hard layers are determined by the Kirchhoff–Love Law; in the soft layers —based on the suppositions—they are linear functions of the local coordinate, perpendicular to the plate (but the normal to the middle surface of the layer does not remain normal after the deformation) and they can be expressed by the displacements of adjacent hard layers.

So, the strain and stress state of the plate will be determined if the displacements of the points belonging to the middle planes of the hard layers—altogether $3n$ functions: $u_\alpha(x, y)$, $v_\alpha(x, y)$, $w_\alpha(x, y)$ —will be known (Fig. 2).

Using the character of linearity of displacement functions by the local coordinates, the displacement fields of hard and soft layers can be built up. From this the strain and curvature vector components will have forms as follows:

for the hard layers:

$$\varepsilon_x^\alpha = u_{\alpha,x}, \quad \varepsilon_y^\alpha = v_{\alpha,y}, \quad \gamma^\alpha = u_{\alpha,y} + v_{\alpha,x}, \kappa_x^\alpha = -w_{\alpha,xx}, \quad \kappa_y^\alpha = -w_{\alpha,yy}, \quad \chi^\alpha = -w_{\alpha,xy}. \tag{5}$$

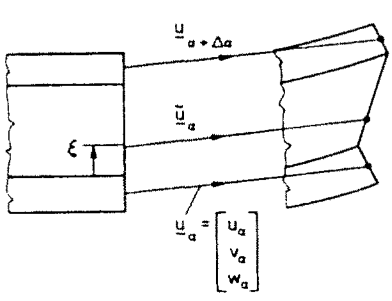

Fig. 2. Displacements of the layers.

for the soft layers:

$$\tilde{\gamma}_{xz}^{\alpha} = \frac{1}{s_\alpha}[u_{\alpha+\Delta\alpha} - u_\alpha + r'_\alpha w_{\alpha,x} + r''_\alpha (w_{\alpha+\Delta\alpha})_{,x}],$$

$$\tilde{\gamma}_{yz}^{\alpha} = \frac{1}{s_\alpha}[v_{\alpha+\Delta\alpha} - v_\alpha + r'_\alpha w_{\alpha,y} + r''_\alpha (w_{\alpha+\Delta\alpha})_{,y}],$$

$$\tilde{\varepsilon}_z^{\alpha} = \frac{1}{s_\alpha}(w_{\alpha+\Delta\alpha} - w_\alpha) \tag{6}$$

where

$$r'_\alpha = \tfrac{1}{2}(h_\alpha + s_\alpha), \quad r''_\alpha = \tfrac{1}{2}(h_{\alpha+\Delta\alpha} + s_\alpha).$$

We can obtain the stress fields of the hard and soft layers with these strain vector components by using the corresponding constitutive equations.

With integration of the deformation energy densities for unit volume given by formulas (2) and using the generalized constitutive equations for *the deformation energy densities for unit surface of the hard and soft layers* we find:

$$\Pi_\alpha = \int_0^{h_\alpha} \mathrm{d}\bar{U}_\alpha \,\mathrm{d}\xi = \tfrac{1}{2}(\underline{\varepsilon}\, \underline{\underline{C}}\, \underline{\varepsilon} + 2\underline{\varepsilon}\, \underline{\underline{K}}\, \underline{\kappa} + \underline{\kappa}\, \underline{\underline{D}}\, \underline{\chi})$$

$$\tilde{\Pi}_\alpha = \int_0^{s_\alpha} \mathrm{d}\tilde{U}_\alpha \,\mathrm{d}\xi = \tfrac{1}{2}(\underline{\tilde{\varepsilon}}\, \underline{\underline{\tilde{C}}}\, \underline{\tilde{\varepsilon}}), \tag{7}$$

or in a more detailed form:

$$\begin{aligned}\Pi_\alpha = \tfrac{1}{2}\{&C_{11}\varepsilon_x^2 + 2C_{12}\varepsilon_x\varepsilon_y + C_{22}\varepsilon_y^2 + C_{66}\gamma^2 \\ &+ 2[K_{11}\varepsilon_x\kappa_x + K_{12}(\varepsilon_x\kappa_y + \varepsilon_y\kappa_x) + K_{22}\varepsilon_y\kappa_y] \\ &+ 4K_{66}\gamma\chi + D_{11}\kappa_x^2 + 2D_{12}\kappa_x\kappa_y + D_{22}\kappa_y^2 + 4D_{66}\chi^2\}_\alpha,\end{aligned}$$

$$\tilde{\Pi}_\alpha = \tfrac{1}{2}s_\alpha^2[B(\tilde{\gamma}_{xz}^2 + \tilde{\gamma}_{yz}^2) + R\tilde{\varepsilon}_z^2]. \tag{8}$$

The density for unit surface of the potential energy of external forces in the case of the stability problem (linear task) has the same form as for an ordinary plate:

$$\Pi_{L\alpha} = \tfrac{1}{2}[N_x^\alpha (w_{\alpha,x})^2 + 2N_{yx}^\alpha w_{\alpha,x} w_{\alpha,y} + N_y^\alpha (w_{\alpha,y})^2]. \tag{9}$$

5. GOVERNING EQUATIONS AND BOUNDARY CONDITIONS OF THE STABILITY PROBLEM

For the mathematical formulation of the stability problem of the multilayered plate as a conservative mechanical system the Trefftz variational principle $[\delta(\delta_*^2 U_0) = 0]$ can be used, where in this case $\delta_*^2 U_0 = I$: the total potential energy of the system written for the state with small disturbances. So, we can get the governing equations and boundary conditions as Euler–Lagrange equations and natural boundary conditions corresponding to the variational principle:

$$\delta I = 0 \tag{10}$$

where

$$I = \sum_{\alpha=-m}^{m} (\bar{U} - L) + \sum_{\alpha=-m}^{m-\Delta\alpha} U \tag{11}$$

is the total potential energy of the plate and

$$\bar{C}_\alpha = \iint \bar{\Pi}_\alpha \, \mathrm{d}x \, \mathrm{d}y, \quad \tilde{C}_\alpha = \iint \tilde{\Pi}_\alpha \, \mathrm{d}x \, \mathrm{d}y, \quad L_\alpha = \iint \Pi_{L\alpha} \, \mathrm{d}x \, \mathrm{d}y.$$

Here the integration has to be extended for the surface of the plate characterized by the half wave lengths λ_x, λ_y in the state of instability.

After rigorous calculations for the governing equations for the stability of a multilayered sandwich-type rectangular flat plate with (constructionally) orthotropic hard layers, we have:

$$\nabla^\alpha_{c1} u_\alpha + \nabla^\alpha_{c3} v_\alpha - (\nabla^\alpha_{k1} w_\alpha)_{,x} + s_\alpha [B_\alpha \tilde{\gamma}^\alpha_{xz} - B_{\alpha-\Delta\alpha} \tilde{\gamma}^{\alpha-\Delta\alpha}_{xz}] = 0,$$

$$\nabla^\alpha_{c2} v_\alpha + \nabla^\alpha_{c3} u_\alpha - (\nabla^\alpha_{k2} w_\alpha)_{,y} + s_\alpha [B_\alpha \tilde{\gamma}^\alpha_{yz} - B_{\alpha-\Delta\alpha} \tilde{\gamma}^{\alpha-\Delta\alpha}_{yz}] = 0,$$

$$\nabla^\alpha_D w_\alpha - (\nabla^\alpha_{k1} u_\alpha)_{,x} - (\nabla^\alpha_{k2} v_\alpha)_{,y} - r'_\alpha s_\alpha B_\alpha (\tilde{\gamma}^\alpha_{xz,x} + \tilde{\gamma}^\alpha_{yz,y})$$
$$- r''_\alpha s_\alpha B_{\alpha-\Delta\alpha} (\tilde{\gamma}^{\alpha-\Delta\alpha}_{xz,x} + \tilde{\gamma}^{\alpha-\Delta\alpha}_{yz,y}) - R_\alpha (w_{\alpha+\Delta\alpha} - w_\alpha) + R_{\alpha-\Delta\alpha} (w_\alpha - w_{\alpha-\Delta\alpha}) + \nabla^\alpha_q w_\alpha = 0, \quad (12)$$

where the operators are as follows:

$$\nabla^\alpha_{c1} = C^\alpha_{11}(\ )_{,xx} + C^\alpha_{66}(\ )_{,yy},$$
$$\nabla^\alpha_{c2} = C^\alpha_{22}(\ )_{,yy} + C^\alpha_{66}(\ )_{,xx},$$
$$\nabla^\alpha_{c3} = (C^\alpha_{12} + C^\alpha_{66})(\ )_{,xy},$$
$$\nabla^\alpha_{k1} = K^\alpha_{11}(\ )_{,xx} + (K^\alpha_{12} + 2K^\alpha_{66})(\ )_{,yy},$$
$$\nabla^\alpha_{k2} = K^\alpha_{22}(\ )_{,yy} + (K^\alpha_{12} + 2K^\alpha_{66})(\ )_{,xx},$$
$$\nabla^\alpha_D = D^\alpha_{11}(\ )_{,xxxx} + 2(D^\alpha_{12} + 2D^\alpha_{66})(\ )_{,xxyy} + D^\alpha_{22}(\ )_{,yyyy},$$
$$\nabla^\alpha_q = [N^\alpha_x(\ )_{,x} + N^\alpha_{xy}(\ )_{,y}]_{,x} + [N^\alpha_y(\ )_{,y} + N^\alpha_{yx}(\ )_{,x}]_{,y},$$

and $\tilde{\gamma}^\alpha_{xz}$, $\tilde{\gamma}^\alpha_{yz}$ have the forms of eqns (6).

The natural boundary conditions—on the boundary $x = \text{const.}$—are:

$$N^\alpha_x = 0, \quad N^\alpha_{xy} = 0, \quad M^\alpha_x = 0, \quad r'_\alpha \tilde{\tau}^\alpha_{xz} + N^\alpha_x w_{\alpha,x} + N^\alpha_{xy} w_{\alpha,y} - (M^\alpha_{x,x} + 2M^\alpha_{xy,y}) = 0. \quad (13)$$

The meaning of the first three conditions is obvious and is well known from the thin plate theory, while the fourth condition states that the generalized transverse shear edge forces on the free boundary must be zero.

Taking into consideration that the facing layers of the plate are hard layers, the corresponding equations for these layers can be taken from eqns (12), replacing zero value for material characteristics of adjacent soft layers, i.e.:

$$\text{at} \quad \alpha = m: \quad B_m = R_m = 0,$$
$$\text{at} \quad \alpha = -m: \quad B_{-m-\Delta\alpha} = R_{-m-\Delta\alpha} = 0. \quad (14)$$

These equations in some cases (for example in the tasks for regularly multilayered plates) play the role of "boundary conditions" perpendicular to the plate direction.

The actual solution of the governing equations (12) shall be obtained with the prevailing boundary conditions. As it is well known from the stability theory of plates, Fourier's method can reasonably be applied to solve the problem of eigenvalue so arisen, provided at least two opposite boundaries are "simply supported" and the plates are only in compression, i.e. $N^\alpha_{xy} = 0$. In cases other than this, whether of more sophisticated boundary conditions or when the plate is also subjected to shear ($N^\alpha_{yx} \neq 0$), other approximations shall be applied. In such cases, it is reasonable to prefer some direct variational method based on expressions (8) and (9) to the solution of eqns (12).

6. INITIAL POSTBUCKLING BEHAVIOUR OF THE PLATE

To investigate the postbuckling behaviour of the multilayered sandwich-type plate for the hard layers the Karman–Timoshenko assumptions and boundary conditions can be used, namely:

(i) the initial postbuckling behaviour of the plate is developed in the instability form;

(ii) the edges of the plate have forms characterized by the linear solution and the opposite edges of the layers can move one to the other; e_x^α, e_y^α are the average relative approaches of the opposite edges of the layers;

(iii) the shear stresses on the edges of the hard layers are zero—at least in integral sense;

(iv) the ratio of normal membrane forces $\bar{n}_\alpha = N_x^\alpha/N_y^\alpha$ remains constant in the post-buckling state of the plate and equal to the ratio $\bar{n}_\alpha$ before buckling;

(v) the normal displacements are comparable to the thickness of anisotropic hard layers.

On the basis of these assumptions the strain tensor components for the hard layers should be taken in nonlinear form; thus for the components of the strain vector we have:

$$\begin{aligned} \varepsilon_x^\alpha &= u_{\alpha,x} + \tfrac{1}{2}(w_{\alpha,x})^2 \\ \varepsilon_y^\alpha &= v_{\alpha,y} + \tfrac{1}{2}(w_{\alpha,y})^2 \\ \gamma^\alpha &= u_{\alpha,y} + v_{\alpha,x} + w_{\alpha,x}w_{\alpha,y}, \end{aligned} \tag{15}$$

but the curvature vector remains in the form of eqn (5) as in the linear task. The strain tensor components for the soft layer remain linear.

The generalized constitutive equations for hard and soft layers also remain the same as in linear task in the form eqns (3) and (4).

The energy densities for the hard and soft layers remain in the form eqns (7) and (8), but the density of the work of external forces should be taken in the form:

$$L_\alpha = -\int_0^b [N_x^\alpha u_\alpha + N_{xy}^\alpha v_\alpha]_0^a \,\mathrm{d}y - \int_0^a [N_{yx}^\alpha u_\alpha + N_y^\alpha v_\alpha]_0^b \,\mathrm{d}x. \tag{16}$$

At the stability problem of compressed plate $N_{xy} = 0$, but in the postbuckling state $N_{xy} = 0$ in integral sense; thus the work of the shear membrane forces should be taken into account in eqn (16).

The solution of the nonlinear problem satisfying the conditions expressing loading and stretching of the edges must now be taken in the forms:

$$u_\alpha = f u_\alpha^* - e_x^\alpha x, \quad v_\alpha = f v_\alpha^* - e_y^\alpha y, \quad w_\alpha = f w_\alpha^*. \tag{17}$$

Here u_α^*, v_α^*, w_α^* are the solutions of the linear stability problem characterizing the displacement field in the moment of instability and f is the deflection amplitude parameter in the postbuckling state of the plate.

The deflection parameter could be determined, for example, by the Ritz direct variational method. By this method instead of the variational problem (10) the total energy of the system could have a minimum, i.e.:

$$\frac{\partial I}{\partial f} = 0, \tag{18}$$

where the total energy of the plate should be calculated by eqn (11) using the nonlinear strain tensor components (15), and eqn (16) for the work of external loads.

From this condition for the deflection parameter f we get a second-order algebraic equation, from which we can determine the actual value of deflection parameters as a function of the system parameters and overloading parameter

$$m = \frac{\bar{N}_y}{\bar{N}_y^*}.$$

Here $\bar{N}_y$ and $\bar{N}_y^*$ are the actual and critical average values, respectively, of external normal forces.

With this parameter f finally we find the solution of the nonlinear initial postbuckling behaviour problem of the orthotropic sandwich plate in the form of eqn (17).

The recommended method gives us the possibility of analysing the stresses in the hard and soft layers and also the amplitude and wave length of bending, depending on the overload parameter, on the different system parameters and on the different values of e_x^α, e_y^α.

7. EXAMPLES: STABILITY AND INITIAL POSTBUCKLING BEHAVIOUR OF REGULARLY MULTILAYERED PLATES

(a) As an example for the solution of the governing equations (12) we may show some results of the investigations of the stability of a regularly multilayered plate based on the papers of Pomázi (1966, 1978).

In the investigation of the stability of a regularly multilayered rectangular plate, all the material, measurement and loading parameters do not depend on α. Let us suppose that on the boundary of the layers Navier-type conditions are valid and each hard layer is loaded with constant N_x, N_y membrane forces only ($N_{xy} \equiv 0$). In this case $\bar{E}$, $\bar{\nu}$, h and E', G, s are the parameters of the hard and soft layers, respectively, and the nonzero stiffness characteristics are:

$$C_{11} = C_{22} = A = \frac{Eh}{1-\bar{\nu}^2}, \quad D_{11} = D_{22} = D = \frac{Eh^3}{12(1-\bar{\nu}^2)}$$

$$B = \frac{\tilde{G}}{s}, \quad R = \frac{\tilde{E}_z}{s} = \frac{E'}{s}\frac{1-\nu}{1-\nu-2\nu_0\nu'}, \quad r = h+s.$$

The Navier-type boundary conditions will be satisfied if, using the Fourier method, the unknown displacement functions are taken in double Fourier series with amplitude functions U_α, V_α, W_α and wave numbers $k_1 = p_1\pi/a = \pi/\lambda_x$, $k_2 = p_2\pi/b = \pi/\lambda_y$ where p_1, p_2 are whole numbers and λ_x, λ_y are the half wave length in the x, y directions at the bending of the hard layers.

Substituting the trial solution functions into the governing equation and into the equations corresponding to the conditions in eqn (14) we get a sixth-order set of ordinary difference equations for the functions U_α, V_α, W_α, with the corresponding boundary conditions.

Looking for these functions in the forms:

$$U_\alpha = U\,e^{\mu\alpha}, \quad V_\alpha = V\,e^{\mu\alpha}, \quad W_\alpha = W\,e^{\mu\alpha} \tag{19}$$

where U, V, W are certain constants and μ is the characteristic exponent, we can formulate the eigenvalue problem for the loading parameter

$$q = \frac{1}{2R}(N_x k_1^2 + N_y k_2^2), \tag{20}$$

and the corresponding characteristic and frequency equations have general forms:

$$\xi^2 + b(q,\rho)\xi + c(q,\rho) = 0, \quad q - C(\rho, m, \bar{\mu}_j) = 0.$$

Here $\xi = \cosh\bar{\mu}$, $\bar{\mu} = \mu\Delta\alpha$, ρ are the system parameters and $\pm\bar{\mu}_j$ ($j = 1, 2$) can be determined as the main values of the complex function $\bar{\mu} = \text{Areacosh}\,\xi$ on the intervallum

$$-\pi \langle \text{area}\,(\xi+\sqrt{\xi^2-1}) \leqslant \pi.$$

The simultaneous solution (by iteration) of the characteristic and frequency equations give a set of critical values of loading parameter q_k as a spectrum of eigenvalues of the boundary problem. It was shown that the number of eigenvalues depends on the form of instability and also on the number of hard layers, and we have:

Instability forms	Number of eigenvalues = n	
antisymmetrical	$\frac{n+1}{2}$	$\frac{n}{2}$
symmetrical	$\frac{n-1}{2}$	$\frac{n}{2}$
n:	even	odd

From the n eigenvalues $n-1$ are in the intervallum

$$q_0 < q_k < q_\Delta \quad (k = 1, \ldots, n-1)$$

where $q_0 = \min q(\xi)$ and $q_\Delta = \max q(\xi)$, determined from the characteristic equation at $-1 \leqslant \xi \leqslant 1$, and the critical value q_M in every case is smaller than q_0.

Using the critical value of characteristic exponents $\bar{\mu}_i^* = \mu_i^*(q_M)$, the relative value of half wave length in the perpendicular direction to the plate can be introduced by the formula:

$$\lambda_i^* = \frac{\pi}{\min |\bar{\mu}_i^*|}. \tag{21}$$

This half wave length characterizes the plate's fields in the moment of instability in the perpendicular direction to the plate. When $\lambda_i^* \ll n$, then the instability form has *local character*, while if $\lambda_i^* \gg n$, then it has *global character*.

The solution in the first case slightly, but in the second case hardly depends on the number of layers.

To illustrate this the stability of a sandwich-type plate with $n = 5$ hard layers was investigated. To get a characterized picture of the forms of instability, for the material characteristics of soft layers, very small values were chosen.

Figure 3 shows the shapes of the critical load function of the hard layers versus the relative half wave length λ_i/h. If the length of the plate $a < \lambda_i^*$, then the form of instability is *global*, but if $a > \lambda_i^*$ then it has *local* character. Of course, to these cases the inequalities $\lambda_i^* \gg n$ or $\lambda_i^* \ll n$ also correspond.

(b) As an example for the initial postbuckling problem we may show the results of the investigations of the initial postbuckling behaviour of the regularly multilayered plate, based on Pomázi (1978, 1982).

In this case the solution of the stability problem has to be taken in real form. Using this form for the displacement functions satisfying the Navier-type boundary conditions we have:

$$\begin{aligned} u_\alpha &= f X_\alpha k_1 \cos k_1 x \sin k_2 y - e_x x, \\ v_\alpha &= f X_\alpha k_2 \sin k_1 x \cos k_2 y - e_y y, \\ w_\alpha &= f Y_\alpha \sin k_1 x \sin k_2 y. \end{aligned} \tag{22}$$

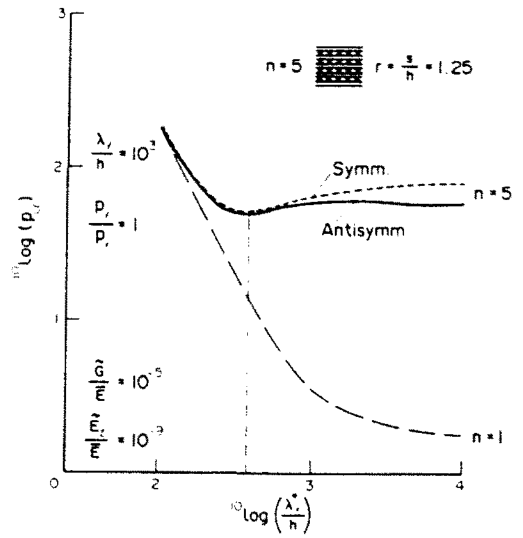

Fig. 3. The shapes of the critical loads versus the half wave lengths.

Here

$$X_x = X_1 + M \cdot X_2$$
$$Y_x = Y_1 + M \cdot Y_2, \tag{23}$$

where

$$X_j = \mathrm{Re}\,(a_j) + \mathrm{Im}\,(a_j), \qquad Y_j = \mathrm{Re}\,(b_j) + \mathrm{Im}\,(b_j)$$
$$a_j = \left(\frac{r}{2}\sinh \bar{\mu}_j\right) \cdot x_j(\alpha), \quad b_j = (1 + \rho_0 - \cosh \bar{\mu}_j) \cdot y_j(\alpha);$$

at symmetric instability

$$x_j(\alpha) = \cosh \mu_j \alpha,$$
$$y_j(\alpha) = \sinh \mu_j \alpha,$$

at antisymmetric instability

$$x_j(\alpha) = \sinh \mu_j \alpha,$$
$$y_j(\alpha) = \cosh \mu_j \alpha,$$

and

$$M = -\frac{M_1}{M_2},$$

where

$$M_j = (1 + 2\rho_0) X_j(m) - X_j(m - \Delta\alpha) + Y_j(m) + Y_j(m - \Delta\alpha) \quad (j = 1, 2).$$

Here all wave numbers μ_j belong to the critical state of the plate (to the critical load parameter q_M) and $\bar{\mu}_j = \mu_j \Delta\alpha$, $r = s + h$ and ρ_0 is the parameter of the system:

$$\rho_0 = \frac{(k_1^2 + k_2^2)A}{2B}.$$

Using Ritz's method the relationship is obtained between the hard layer's deflection parameter and the plate-end shortening for each hard layer of the rectangular plate with Navier's type boundary conditions. Numerical analysis shows the characteristic effect of

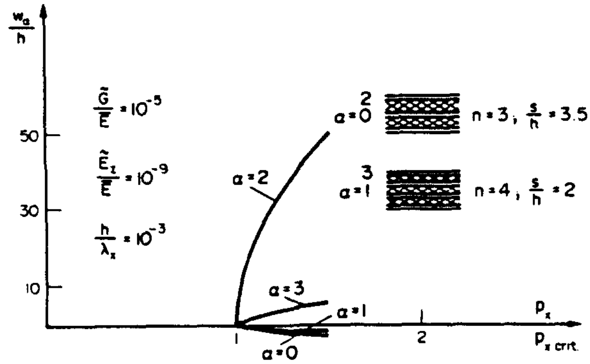

Fig. 4. Developing of the deflection parameters versus the overloading.

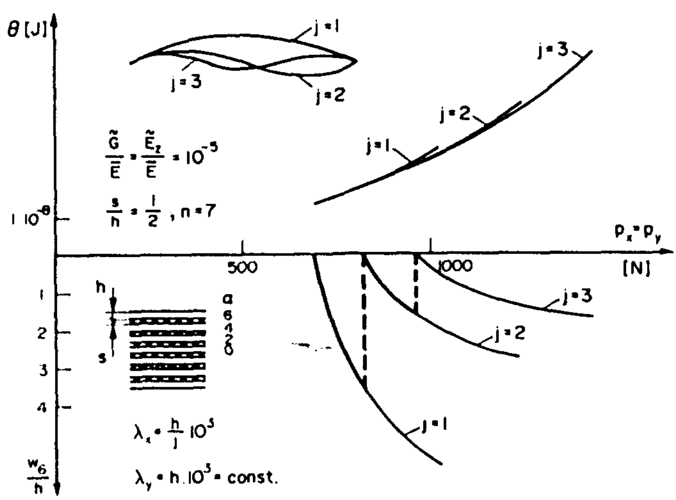

Fig. 5. The total energy and the deflection parameter's functions versus the compressive load.

the antiplane normal strain on the "local" and "global" instability forms and the developing of these in the postbuckling state.

For each hard layer we get a different value of the deflection parameter w_z/h, depending on the system parameters, the character of instability (local or global) and the form of instability (symmetric or antisymmetric).

Figure 4 shows the shapes of deflection of the parameter's function for different hard layers in layered plates with $n = 3$ and $n = 4$ hard layers at antisymmetrical forms of instabiity. It can be seen that the character of instability is local because each neighbouring hard layer is deflected to the opposite side.

For a plate with $n = 7$ hard layers Fig. 5 shows the curves of the deflection parameters and the corresponding total energy functions versus the compressive load p_x, acting on the hard layers. Visibly the minimum of total energy belongs to different modes (to different wave lengths λ_x) and therefore only these modes could be realized—which shows the possibility of *mode jumping* in the postbuckling state of the plate.

8. CONCLUDING REMARKS

The given governing equations and the results of the analytical and approximate solutions for the stability and initial postbuckling behaviour of multilayered sandwich-type plates can be used as a "control task" in the investigations of the stability of layered composites.

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